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On the analogy between thermal and rotational hydrodynamic stability

By WALTER R. DEBLER
Department of Engineering Mechanics, The University of Michigan

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The correspondence between the eigenvalues for the problem of the onset of convection in a fluid confined between two horizontal plates and for the stability of viscous flow between two cylinders rotating at almost the same angular velocity has been known for some time. The recent work of Chandrasekhar (1961) has prompted the extension of the analogy to a larger group of rotating cylinder problems and their associated convection cases in which the primary temperature distribution is parabolic. This paper shows the analogy between these two problems and presents data which give the corresponding temperature distribution for a given ratio of angular velocities between the two cylinders. The equivalent Rayleigh numbers are listed for the Taylor numbers given by Chandrasekhar (1954). The eigenfunctions for several of the parabolic temperature profiles are determined. These results show that the single vortex convection pattern becomes a double vortex for certain initial temperature distributions. The critical Rayleigh numbers for the stability of a layer of water which is near 4 °C is also found by analogy.

1. Introduction

The classical problem of convective motion in a fluid heated from below (e.g. Pellew & Southwell 1940) has its counterpart in the secondary flow within a narrow annulus between two coaxial cylinders which are rotated at almost the same angular velocity (e.g. Taylor 1923). Two papers by Chandrasekhar (1954, 1961) provide the basis for extending the analogy to other thermal and rotational cases. It is the purpose of this paper to point out how the data given by Chandrasekhar (1954) can be applied to the stability of a horizontal layer of fluid with homogeneously distributed heat sources and the thermal stability of water at 4 °C.

Both of the latter problems were treated by Debler (1959) using a method given by Chandrasekhar (1954). Various boundary conditions were examined and some of the answers anticipate a few of the results of Sparrow, Goldstein & Jonsson (1964). The similarity of the form of the differential equations and boundary conditions treated by Chandrasekhar and Debler, and the curious correspondence between the results of the two problems suggested that the two systems of differential equations may have been interrelated. The explanation for the seemingly similar results is contained in Chandrasekhar (1961). The work of Davey (1962) contains a function which is the solution of a differential equation that is adjoint to one which arises initially out of his treatment of the Taylor vortices. This function and the adjoint differential equation are equivalent to those coming from one of the cases considered in this paper.

2. Governing equations

Consider a horizontal layer of fluid of depth b, confined between two parallel planes $x_3 = 0$ and $x_3 = b$. These planes are solid, heat conducting and of constant temperature. The fluid has uniformly distributed heat sources. The equations of motion, heat conduction and mass conservation that are applicable to this problem are

$$\rho\left(\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}\right) = -\delta_{i3}g\rho-\frac{\partial}{\partial x_i}\left(p-(\lambda+\mu)\frac{\partial u_j}{\partial x_j}\right)+\rho\nu\Delta u_i, \tag{1}$$

$$\frac{1}{\kappa}\frac{D\theta}{Dt}-\frac{q}{k} = \Delta\theta \tag{2}$$

and

$$\partial\rho/\partial t+\partial(\rho u_j)/\partial x_j = 0, \tag{3}$$

in which the subscripts in the Navier–Stokes equations have values of 1, 2, and 3 for the co-ordinate directions, and the repetition of a particular suffix implies the summation convention of tensor analysis; u_i is the component of the fluid velocity in the ith co-ordinate direction; ρ is the mass density of the fluid; λ a Lamé constant; g the gravitational constant; p the pressure in the fluid; μ and ν the dynamic and kinematic viscosities; δ_{i3} the Kronecker delta, being zero for $i \neq 3$ and unity for $i = 3$; and Δ the symbol for the Laplacian operator. In

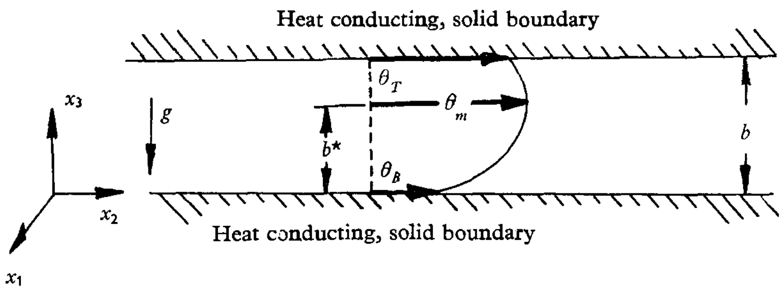

FIGURE 1. Co-ordinate system and mean temperature distribution in fluid layer.

writing equation (2) the dissipation of mechanical energy into heat is considered small and therefore neglected. Also, in the foregoing equations θ is the absolute temperature of the fluid, κ is the thermal diffusivity; q is the time rate of heat generation per unit volume of fluid, k is the thermal conductivity, and D/Dt is the substantial derivative

$$D/Dt = \partial/\partial t+u_j\,\partial/\partial x_j.$$

The co-ordinate system employed for this problem is presented in figure 1 along with the mean temperature distribution in the fluid layer.

The density of the fluid can be expressed as

$$\rho = \rho_m\{1-\alpha(\theta-\theta_m)\} \tag{4}$$

if the temperature differences in the fluid are small. In equation (4) ρ_m is the density at the maximum temperature, θ_m; and α is the coefficient of thermal expansion.

Under equilibrium conditions the fluid layer is quiescent and the heat liberated by the fluid is carried to the boundaries by conduction alone. Accordingly, equations (1) and (2) can be solved for this condition and by assuming constant thermal properties one has

$$\bar{u}_i = 0, \tag{5}$$

$$\bar{\theta} = (q/2k)(bx_3 - x_3^2) + (x_3/b)(\theta_T - \theta_B) + \theta_B, \tag{6}$$

$$d\bar{\theta}/dx_3 = (\theta_m - \theta_T)(2b^* - 2x_3)/(b - b^*)^2, \tag{7}$$

$$(\theta_T - \theta_B)/(2bb^* - b^2) = q/2k = (\theta_m - \theta_T)/(b - b^*)^2, \tag{8}$$

$$\partial\bar{p}/\partial x_i = -\delta_{i3}(g\bar{\rho}), \tag{9}$$

in which the bar over a symbol denotes a mean quantity, and hence the value associated with the quiescent state. θ_m is the maximum temperature of the fluid layer while pure conduction is taking place, and θ_T and θ_B are the temperatures of the upper and lower boundaries of the fluid, respectively. It will be assumed that the boundary temperatures will remain constant and the rate of heat generation will remain the same regardless of the motion of the fluid.

Now consider small departures in the values of the temperature, pressure, density, and velocity from those existing during the conduction state. The values of these quantities will be

$$\theta = \bar{\theta} + \theta', \quad p = \bar{p} + p', \quad \rho = \bar{\rho} + \rho', \quad u_i = u_i', \tag{10}$$

if a linear perturbation is employed, and the primed quantities are these perturbations.

The perturbed variables (10) are then substituted into equations (1) and (2) and equations result from which a solution for the perturbations can be obtained. The solution will be examined to determine the conditions under which a perturbed quantity can have a steady-state value other than zero. The equation resulting from the substitution of (10) into equation (1) is

$$\frac{\partial u_i}{\partial t} = -\delta_{i3}\left(g\frac{\rho'}{\bar{\rho}}\right) - \frac{1}{\bar{\rho}}\frac{\partial}{\partial x_i}\left(p' - (\lambda + \mu)\frac{\partial u_j}{\partial x_j}\right) + \nu\Delta u_i. \tag{11}$$

In writing this equation the primed quantities are considered to be small and all products of perturbation quantities can be ignored because their magnitude will be very much smaller in comparison with the other terms in the equation. The result of this simplification is three scalar equations which are linear.

The coefficient $1/\bar{\rho}$ is approximated by $1/\rho_m$ and it follows also that

$$\rho' = \rho_m(-\alpha\theta').$$

By properly differentiating equation (11) the pressure terms can be eliminated with the result that

$$\frac{\partial}{\partial t}\left\{\Delta u_3 - \frac{\partial}{\partial x_3}\left(\frac{\partial u_i}{\partial x_i}\right)\right\} = g\alpha\Delta_{12}\theta' + \nu\Delta\left\{\Delta u_3 - \frac{\partial}{\partial x_3}\left(\frac{\partial u_i}{\partial x_i}\right)\right\}, \tag{12}$$

in which

$$\Delta_{12} = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2.$$

The equation of continuity provides the justification for neglecting

$$\partial(\partial u_i/\partial x_i)/\partial x_3$$

in comparison with Δu_3. With such an assumption (i.e. the Boussinesq approximation) the last equation reduces to

$$(\partial/\partial t - \nu\Delta)\,\Delta u_3 = g\alpha\Delta_{12}\theta'. \tag{13}$$

A similar development can be applied to the energy equation so that it can be written in the form

$$\frac{1}{\kappa}\left(\frac{\partial\theta'}{\partial t} + u_i\frac{\partial\overline{\theta}}{\partial x_i}\right) = \Delta\theta'. \tag{14}$$

The boundary conditions for these equations are

$$\theta' = 0 = u_3 = \partial u_3/\partial x_3 \quad \text{at} \quad x_3 = 0, b. \tag{15}$$

It is convenient to non-dimensionalize the above equations and boundary conditions prior to attempting a solution. A suitable means of doing this is by setting

$$\left.\begin{aligned} &\tau = t\kappa/b^2, \quad (x, y, z) = (x_1/b,\, x_2/b,\, x_3/b),\\ &w = u_3 b/\kappa, \quad T = \theta'/(\theta_m - \theta_T). \end{aligned}\right\} \tag{16}$$

The resulting equations are solved by assuming a solution of the form

$$w = f\hat{w}e^{\sigma\tau}, \tag{17}$$

$$T = f\hat{T}e^{\sigma\tau}, \tag{18}$$

in which f is a function of x and y only, the functions $\hat{w}$ and $\hat{T}$ are dependent only on z, and σ has real and imaginary parts σ_r and σ_i, respectively. For σ_r greater than zero the solution (representing the disturbance) will grow exponentially with time, and for σ_r less than zero, the perturbation will decay. If σ_i is non-zero the perturbation quantities will be oscillatory.

The final equations to be solved are

$$\partial^2 f/\partial x^2 + \partial^2 f/\partial y^2 + a^2 f = 0, \tag{19}$$

$$[\sigma - (D^2 - a^2)]\,\hat{T} = -\hat{w}(2\eta - 2z)/(1-\eta)^2, \tag{20}$$

$$[\kappa\sigma/\nu - (D^2 - a^2)]\,[D^2 - a^2]\,\hat{w} = -\hat{T}a^2 R, \tag{21}$$

in which a is the cell number,

$$\eta = b^*/b, \quad D = d/dz,$$

and

$$R = (g\alpha b^3/\nu\kappa)\,(\theta_m - \theta_T). \tag{22}$$

The boundary conditions are

$$\hat{w} = 0 = D\hat{w} = \hat{T} \quad \text{at} \quad z = 0, 1. \tag{23}$$

For this problem the 'principle of exchange of stabilities' will be invoked so that at neutral stability the real and imaginary parts of σ are both taken as being zero.

3. Method of solution

The thermal boundary conditions will be satisfied by a function $\hat{T}$ having a Fourier series expansion of the form

$$\hat{T} = \sum_{n=1}^{\infty} A_n \sin n\pi z \quad (n = 1, 2, 3, \ldots). \tag{24}$$

This expression can be introduced into equation (21) with the result that

$$\hat{w} = a^2 R \sum_{n=1}^{\infty} A_n \left\{ B_n \cosh az + C_n \sinh az + D_n z \sinh az + E_n z \cosh az + \frac{1}{N_n^2} \sin n\pi z \right\} \tag{25}$$

in which $$N_n = (n\pi)^2 + a^2.$$

The boundary conditions on $\hat{w}$ are sufficient to specify B_n, C_n, D_n and E_n. The series equivalents for $\hat{T}$ and $\hat{w}$ can now be substituted into equation (20) yielding

$$\sum_{n=1}^{\infty} A_n(-N_n) \sin n\pi z = \frac{2a^2 R}{(1-\eta)^2} \sum_{n=1}^{\infty} A_n \left\{ C_n(\eta - z) \sinh az + D_n(\eta z - z^2) \sinh az + E_n(\eta z - z^2) \cosh az + \frac{(\eta z - z^2)}{N_n^2} \sin n\pi z \right\}. \tag{26}$$

If this equation is to be true, the coefficients of $\sin n\pi z$ on both sides of the equation must be equal for all n. The method for equating these coefficients is the usual one of Fourier series (i.e. multiplying both sides of equation (26) by $\sin m\pi z$ and integrating from zero to one, the range of z). This results in a set of m equations in n unknowns, A_n. The infinite determinant of the coefficients of the A_n's in these equations must be zero for a non-trivial solution to exist, so that

$$\left| \frac{-N_n(1-\eta)^2 \delta_{mn}}{4a^2 R} - (m/n) \right| = 0, \tag{27}$$

in which

$$(m/n) = \int_0^1 \left\{ C_n(\eta - z) \sinh az + D_n(\eta z - z^2) \sinh az + E_n(\eta z - z^2) \cosh az + \frac{(\eta z - z^2)}{N_n^2} \sin n\pi z \right\} \sin m\pi z \, dz. \tag{28}$$

The solution of equation (27) can be obtained by solving $n \times n$ determinants of finite size and noting that the solution converges rapidly for increasing values of n. Indeed Chandrasekhar (1954) shows that very good approximations are obtained for $n = 1$. In considering the case for which the maximum temperature occurs at the midplane Debler (1959) found that (m/n) vanished for $m = n$. Hence a 2×2 determinant had to be solved for the first approximation. Nevertheless the method of solution is straightforward and easy.

4. Analogous differential system

The equations presented in the last section could now be solved for the various values of the parameters. This appears to be unnecessary in view of the correspondence between these equations and those treated in detail by Chandrasekhar (1954). To expedite the demonstration of the equivalence of the two systems of equations and boundary conditions, equations (20) and (21) with $\sigma = 0$ are rewritten as

$$(D^2-a^2)^2 W' = \psi', \tag{29}$$

$$(D^2-a^2)\psi' = -W'\lambda'(1+\alpha'\zeta), \tag{30}$$

$$W' = 0 = DW' = \psi' \quad \text{at} \quad \zeta = 0, 1, \tag{31}$$

by introducing $\zeta = 1-z, \quad \psi' = a^2R\hat{T}, \quad W' = \hat{w},$

and $\alpha' = 1/(\eta-1), \quad \lambda' = -2\alpha'(a^2R).$

The equations governing the motion of the flow between two concentric cylinders with small spacing can be written, using the notation of Chandrasekhar (1954), as

$$(D^2-a^2)^2 W = (1+\alpha'\zeta)\psi \tag{32}$$

and

$$(D^2-a^2)\psi = -W\lambda, \quad \lambda = a^2T, \tag{33}$$

with

$$W = DW = \psi = 0 \quad \text{for} \quad \zeta = 0 \quad \text{and} \quad 1. \tag{34}$$

The similarity in form between equations (32) and (33) and (29) and (30) is immediately apparent. In a subsequent work Chandrasekhar (1961) shows implicitly that the differential systems (29)–(31) and (32)–(34) have the same eigenvalues since the matrix elements (cf. equation (27)) for the two systems are transposes of one another.

5. Results

In view of what has been said, it is now possible to extend the applicability of the results of Chandrasekhar (1954). The appropriate data of Sparrow, *et al.* (1964) are not included in table 1 since their results by direct methods are in complete agreement with the values obtained by analogy if one keeps in mind the definition of R. The appropriate values of their parameter N_s are given, however, for ease of comparison.

All of the values of R given in the table can be converted to a Rayleigh number based on $(\theta_B-\theta_T)$ by using equation (8). This substitution of $(\theta_B-\theta_T)$ for $(\theta_m-\theta_T)$ seems to be very desirable for the second and third entries of column 5 which correspond to cases for which the highest fluid temperature is at the lower boundary but for which $d\theta/dx_3$ is not zero at this point. Consequently, θ_m does not occur within the fluid layer. The first entry in column 5 has been left blank because a direct application of the analogue equations would yield a meaningless result. This situation arises out of the way the differential equations were non-dimensionalized. Indeed if the temperature difference in equation (22) is replaced by its equivalent from equation (8) one has

$$R = R'(b-b^*)^2/(b^2-2bb^*) = R'(1-\eta)^2/(1-2\eta)$$

in which $R' = g\alpha b^3(\theta_B-\theta_T)/\nu\kappa$

and λ' in equation (30) is

$$\lambda' = -2\left(\frac{1}{\eta-1}\right)a^2\frac{(1-\eta)^2}{(1-2\eta)}R' = R'(2a^2)\left(\frac{1-\eta}{1-2\eta}\right).$$

For $\eta \to -\infty$, $\lambda' \to a^2R'$ and $R' \to 1708$. The second and third entries become, in terms of R', $\frac{3}{4}T = 1706$ and $\frac{5}{8}T = 1703$, respectively.

Chandrasekhar (1954)			Analogy		
α'	a	T	$\eta = \frac{b^*}{b} = 1+\frac{1}{\alpha'}$	$R = \frac{-T}{2\alpha'}$	$N_s = \frac{1}{1-2\eta}$
0	3·12	1,708·0	$-\infty$	—	0
−0·5	3·12	2,275·3	−1·0	2,275	0·33
−0·75	3·12	2,725·3	−0·333	1,817	0·60
−1·0	3·12	3,390·3	0	1,695	1
−1·25	3·13	4,462·5	0·200	1,785	1·67
−1·50	3·20	6,417·1	0·333	2,139	3
−1·60	3·25	7,687·7	0·375	2,402	4
−1·70	3·34	9,432·6	0·412	2,774	5·69
−1·80	3·50	11,820	0·445	3,283	9·1
−1·90	3·70	14,943	0·474	3,932	19·45
−1·95	3·86	16,764	0·487	4,298	38·45
−2·00	4·00	18,677	0·500	4,669	∞
−2·25	4·60	30,458	0·556	6,768	− 8·93
−2·50	5·05	46,192	0·600	9,238	− 5
−2·75	5·60	67,592	0·636	12,289	− 3·69
−3·00	6·05	95,625	0·667	15,938	− 3

TABLE 1. Equivalent Taylor and Rayleigh numbers

The value of R for $\eta = 0$ is associated with a temperature profile which has a derivative of zero at $\eta = 0$. This implies that there is zero heat transfer at this boundary under the unperturbed conditions. The boundary conditions which were applied to all of the problems whose solutions are tabulated in the table, state that at the onset of convection the temperature at the boundary does not change. However, the local heat transfer may be affected by the convection.

If instead of applying the isothermal boundary condition at the lower boundary for the convection problem corresponding to $\alpha' = -1$, an adiabatic condition were specified (e.g. $D\hat{T} = 0$ instead of $\hat{T} = 0$, at $z = 0$, all other conditions the same) then the critical Rayleigh number is 1393 according to Debler (1959) at a cell number of 2·5.

6. Eigenfunction

Although the eigenvalues of the two sets of equations given by (29)–(31) and (32)–(34) have been shown to be the same, the associated eigenfunctions which specify the perturbation velocities are different. Taylor (1923) carried out some calculations for the equations given by (32)–(34) and found that when the ratio of the angular velocities between the outer and inner cylinders was −1·5 (i.e. $\mu = -1{\cdot}5 = \alpha' + 1$, $\alpha' = -2{\cdot}5$) the eigenfunction corresponding to the radial velocity was zero at a point in the gap between the two cylinders.

This would mean that the fluid nearest the inner cylinder would not move into the region nearest the outer cylinder and that a double set of vortex rings would be generated. Taylor's (1923) experiments which bracket the value of $\alpha' = -2{\cdot}5$ and include $\alpha' = -2{\cdot}0$ confirm this prediction.

Debler (1959) determined the eigenfunction for the problem having $\eta = 0{\cdot}5$ which corresponds to $\alpha' = -2{\cdot}0$. His calculations showed that the velocity eigenfunction was zero only at the boundaries. Hence, one would expect that when convection occurs the fluid moves from the bottom plate to the upper one and then back down (i.e. a 'single vortex'). This motion was observed in the three experiments which were conducted.

These experiments were performed by creating the temperature distribution in a layer of water through Joule heating with an alternating current. The fluid was confined between horizontal copper plates which acted as both the electrodes and the isothermal surfaces. A small amount of copper sulphate was added to the water to obtain practical resistance values. The experiments, preliminary in nature, yielded critical Rayleigh numbers which were higher than the 4669 predicted by the analysis. But they did show that the resulting convection was a single vortex. Minute drops of a neutrally buoyant oil were injected into the layer during an experiment. If the droplets had no subsequent motion the heat dissipation was increased. At some power setting the droplets began moving in a path which took them from one plate to the other. After the necessary observations had been made the power was slowly reduced until the motion of the droplets stopped. The critical Rayleigh numbers were determined by averaging the power settings for the observations under which the motion commenced and ceased.

Despite the fact that Taylor's (1923) and Debler's (1959) experiments showed markedly different convection patterns for the same value of α' it was believed that if the point of maximum temperature, η, were to be sufficiently high above the midplane, the fluid would not move from a small unstable layer into a large stable one. Yet convection should occur and would no doubt take the form of a double vortex or cell.

A series of calculations were initiated to determine the eigenfunctions for various values of η. Below $\eta = 0{\cdot}546$ the velocity eigenfunction is always of a single sign and above this value of η the eigenfunction changes sign in the interval. In order to establish the value of η which would mark the change in the character of the convection, the second derivative, evaluated at the lower boundary, was examined for the various eigenfunctions. In view of the conditions demanded by (23), the value of η was varied until the second derivative was zero.

This demarcation value of η is very near to 0·55, which corresponds to the value of N_s associated with the maximum value $\tilde{\mathfrak{R}}$ tabulated by Sparrow, Goldstein & Jonsson (1964). Indeed, it was thought possible that the vanishing of the second derivative at the lower boundary would occur when $\tilde{\mathfrak{R}}$ was maximum. The calculations which were performed to determine the critical values of a

and R in the neighbourhood of $\eta = 0{\cdot}546$ did not yield a relationship between η and the critical R, from which any deductions could be made. However, this difficulty could have arisen from the degree of accuracy with which the numerical solution of the characteristic equation was performed. For $\eta = 0{\cdot}546$ the minimum values of R for $a = 4{\cdot}51$ were 6463·4, 6360·7, and 6348·1 when determinants (cf. equation (27)) were used with $n = 2$, 3 and 4, respectively. A more precise determination of the minimum R for various η indeed may show that $\mathfrak{R}$ reaches its maximum when the type of convection pattern changes.

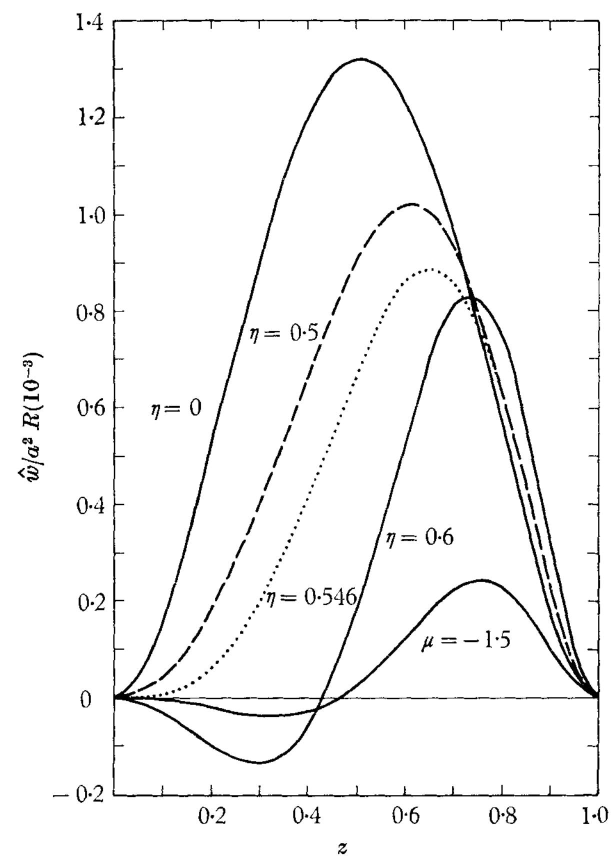

FIGURE 2. Eigenfunction $\hat{w}$ on an arbitrary scale, for various temperature distributions. Curve for $\mu = -1{\cdot}5$ from Taylor (1923), plotted with $z = 1-y$, from comparison with $\eta = 0{\cdot}6$.

Figure 2 gives the velocity eigenfunctions, plotted on an arbitrary scale, for some of the interesting cases which were examined. The curve for $\eta = 0$ could have been obtained from the work of Davey (1962) who treated several problems of Taylor vortices by studying the associated adjoint functions which, as is now apparent, are exactly the eigenfunctions for the Rayleigh problems in which the temperature profile is parabolic. In relating the work of Davey (1962) to the present one it is necessary to employ the co-ordinate transformation $z = \frac{1}{2} - x$.

This figure also presents a curve showing the eigenfunction for the case which is analogous to $\eta = 0{\cdot}6$ and was determined by Taylor (1923) whose independent variable, y, has been plotted as z through the change of co-ordinate $z = 1-y$. These co-ordinate transformations are similar to the one employed writing

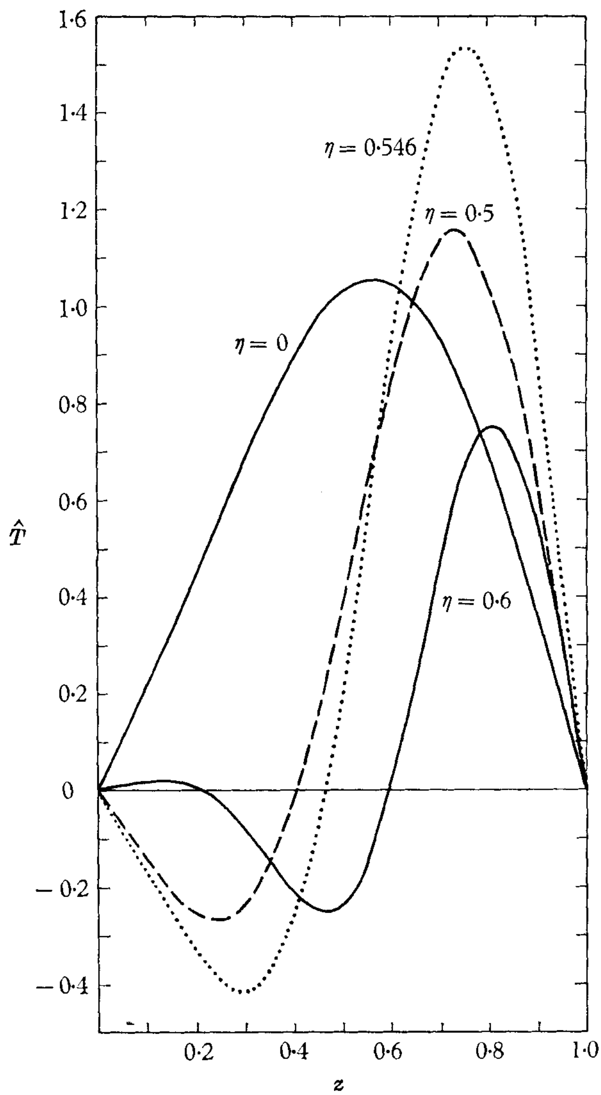

FIGURE 3. Eigenfunction $\hat{T}$ on an arbitrary scale for various temperature distributions given by η.

equation (30) and are necessary to give the same value for the independent variable when discussing 'stable' and 'unstable' regions of the fluid layer or gap. Indeed for the thermal problems under discussion the most stable temperature gradient occurs for the least values of z but for the problems of Davey (1962) and Taylor (1923) the least stable region is located at the smallest values of x and y, respectively.

The temperature eigenfunctions are shown in figure 3. In order to determine these functions it was necessary to take a great many terms because of the slow decrease of the A_n which are necessary in equation (24). The first 20 A_n's are given in table 2 as determined with the aid of a digital computer and a simultaneous linear equation subroutine. These values may be helpful to persons working on related problems. The coefficients in equation (25), for the boundary conditions (23) are also given here in order to completely specify the eigenfunctions. Thus,

$$B_n = 0, \quad C_n = \frac{n\pi}{N_n^2}\left[\frac{a+(-1)^n \sinh a}{\sinh^2 a - a^2}\right],$$

$$D_n = -C_n[1+(-1)^{n+1}\cosh a], \quad E_n = C_n[(-1)^{n+1}\sinh a].$$

$\eta =$	0	0·5	0·546	0·6
A_1	1·000000	1·000000	1·000000	1·000000
A_2	−0·128095	−1·393589	−1·909101	−3·301730
A_3	−0·027394	0·219464	0·582551	3·286538
A_4	0·005611	0·051204	0·036684	−0·810999
A_5	−0·002781	−0·016627	−0·034630	−0·121270
A_6	0·001016	0·012728	0·020136	0·079184
A_7	−0·000552	−0·004428	−0·010184	−0·067351
A_8	0·000267	0·003513	0·005870	0·028881
A_9	−0·000162	−0·001423	−0·003357	−0·023347
A_{10}	0·000092	0·001230	0·002098	0·010916
A_{11}	−0·000060	−0·000553	−0·001319	−0·009348
A_{12}	0·000038	0·000512	0·000882	0·004712
A_{13}	−0·000026	−0·000248	−0·000595	−0·004257
A_{14}	0·000018	0·000242	0·000420	0·002273
A_{15}	−0·000013	−0·000124	−0·000298	−0·002144
A_{16}	0·000009	0·000126	0·000219	0·001197
A_{17}	−0·000007	−0·000067	−0·000162	−0·001170
A_{18}	0·000005	0·000070	0·000123	0·000677
A_{19}	−0·000004	−0·000039	−0·000094	−0·000680
A_{20}	0·000003	0·000042	0·000073	0·000405

TABLE 2. Ratio of Fourier coefficients to A_1 for various temperature distributions

7. Water at 4 °C

The density characteristics of water near 4 °C give rise to a stability problem which resembles those just discussed. In the range between 0 and 8 °C the temperature-density relationship for water can be closely approximated by

$$(\theta-4)^2 = -1{\cdot}25(10^5)\,(\rho - 0{\cdot}999973).$$

If one has a layer of water in which the temperature of 4 °C occurs at some elevation and which is situated between two rigid, horizontal, isothermal boundaries separated by a distance b, the governing equations for the disturbances which result are

$$(D^2-a^2)^2\hat{w} = (1-z/\eta^*)\,\tilde{T} \tag{35}$$

and

$$(D^2-a^2)\,\tilde{T} = -\hat{w}(2a^2C\eta^*) \tag{36}$$

for

$$\hat{w} = 0 = D\hat{w} = \tilde{T} \quad \text{at} \quad z = 0, 1. \tag{37}$$

The co-ordinate system is the same as that in figure 1; however, for the situation now under discussion there is a linear temperature gradient given by

$$\theta = 4^\circ - (x_3 - \eta^* b)\,\beta$$

in which

$$\beta = (\theta_B - 4)/\eta^* b = (\theta_B - \theta_T)/b$$

and $\eta^* b$ is the elevation above the lower surface at which 4 °C occurs. The characteristic parameter C is given by in

$$C = g\beta^2 b^5/\kappa\mu\omega$$

in which the terms in the denominator are evaluated at 4 °C and

$$\omega = 1{\cdot}25\,(10^5)\,(^\circ\mathrm{C})^2\,\mathrm{cm}^3/\mathrm{g}.$$

The dependent variables are related to the perturbed temperature and velocity (cf. equation (10)) by

$$\theta' = (\theta_B - \theta_T)\,\hat{T} f e^{\sigma\tau} = \{(\theta_B - \theta_T)/2a^2 C\eta^*\}\,\tilde{T} f e^{\sigma\tau}$$

and

$$u_3 = (\kappa/b)\,\hat{w} f e^{\sigma\tau}$$

by means of a series of steps involving non-dimensionalization and separation of variables similar to those given in § 2.

The linear variation in the temperature in the undisturbed state would be the case for heat conduction across the fluid layer, provided that k is constant. Because the temperature gradient in the fluid, β, occurs as the square in the parameter C it is immaterial in which direction the heat is flowing.

The form of equations (35) and (36) can be compared with equations (32) and (33) to obtain the critical parameters from table 1 or other sources. For example, table 1 gives the solution for $\alpha' = -1$ as being $a = 3{\cdot}12$ and $T = 3390{\cdot}3$. Thus for $\eta^* = 1$, $a = 3{\cdot}12$ and $2C(1) = 3390{\cdot}3$ or $C = 1695$.

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